TESTE MODELO TMII 2020

1. a) f diferenciável $\Rightarrow$ f contínua logo $f(-1,2) = \lim_{(x,y)\to(-1,2)} f(x,y) = 5.$

b) $z = f(-1,2) + \frac{\partial f}{\partial x}(-1,2)(x+1) + \frac{\partial f}{\partial y}(-1,2)(y-2)$

$z = 1 + 3(x+1) - 2(y-2)$

c) $f(-0.9, 2.2) \approx 1 + 3(-0.9+1) - 2(2.2-2) = 0.3$

d) $f(-0.9, 2.2) \approx 0.3 + \frac{1}{2}\begin{bmatrix} -0.1 & 0.2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -0.1 \\ 0.2 \end{bmatrix}$

$= 0.3 + \frac{1}{2}\begin{bmatrix} -0.1 & 0.2 \end{bmatrix}\begin{bmatrix} -0.3 \\ 0.5 \end{bmatrix}$

$= 0.3 + \frac{1}{2}\, 0.13 = 0.365$

2 a) $\lim_{\substack{(x,y)\to(0,0) \\ y=mx}} g(x,y) = \lim_{x\to 0} \frac{2x^3 - (mx)^2}{x^2 + (mx)^2} = \lim_{x\to 0} \frac{\cancel{x^2}(2x - m^2)}{\cancel{x^2}(1+m^2)} = \frac{-m^2}{1+m^2}$

Como os limites anteriores dependem de $m \in \mathbb{R}$, g não tem limite quando $(x,y) \to (0,0)$ e, portanto, g não é contínua em $(0,0)$.

b) Para $(x,y) = (0,0)$

$\frac{\partial g}{\partial y}(0,0) = \lim_{h\to 0} \frac{g(0,0+h) - g(0,0)}{h} = \lim_{h\to 0} \frac{-1-0}{h} = \infty,$

portanto $\frac{\partial g}{\partial y}$ não está bem definida na origem.

Para $(x,y) \neq (0,0)$

$\frac{\partial g}{\partial y} = \left(\frac{2x^3 - y^2}{x^2+y^2}\right)'_y = \frac{-2y(x^2+y^2) - 2y(2x^3-y^2)}{(x^2+y^2)^2} = \frac{-2y(2x^3+x^2)}{(x^2+y^2)^2}$

$D_{\frac{\partial g}{\partial y}} = \mathbb{R}^2 \setminus \{(0,0)\}.$

c) $\frac{\partial g}{\partial x}(0,0) = \lim_{h \to 0} \frac{g(0+h,0) - g(0,0)}{h} = \lim_{h\to 0} \frac{1}{h} \cdot \frac{2h^2}{h^2} = 2.$

d) g não é diferenciável em $(0,0)$ porque não é contínua em $(0,0)$.
g é diferenciável em $\mathbb{R}^2 \setminus \{(0,0)\}$ pq é o quociente de duas funções polinomiais.

3

a) $\frac{\partial f}{\partial x} = e^y + y\cos(xy) \Rightarrow \frac{\partial f}{\partial x}(2,0) = 1$

$\frac{\partial f}{\partial y} = xe^y + x\cos(xy) \Rightarrow \frac{\partial f}{\partial y}(2,0) = 4$

$f'_{\vec{v}}(2,0) = \vec{\nabla} f(2,0) \cdot \vec{v} = (1,4)\cdot(-3,2) = 5$

b) $\vec{\nabla} f(2,0) = (1,4)$

c) $\vec{x}_1 = \vec{x}_0 - h \cdot \vec{\nabla} f(\vec{x}_0) = (2,0) - 0.1(1,4) = (1.9, -0.4)$

4 Escrevendo $u = 2x - y$ e $v = 2x + y$, obtemos

a) $\frac{\partial w}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x} + \frac{dg}{dv}\frac{\partial v}{\partial x} = 2\frac{df}{du} + 2\frac{dg}{dv}$

$\frac{\partial w}{\partial y} = \frac{df}{du}\frac{\partial u}{\partial y} + \frac{dg}{dv}\frac{\partial v}{\partial y} = -\frac{df}{du} + \frac{dg}{dv}$

b) $\frac{\partial^2 w}{\partial x^2} = 2\frac{\partial}{\partial x}\left(\frac{df}{du}\right) + 2\frac{\partial}{\partial x}\left(\frac{dg}{dv}\right)$

$= 2\frac{d}{du}\left(\frac{df}{du}\right)\frac{\partial u}{\partial x} + 2\frac{d}{dv}\left(\frac{dg}{dv}\right)\frac{dv}{dy}$

$= 4\frac{d^2 f}{du^2} + 4\frac{d^2 g}{dv^2}$

$\frac{\partial^2 w}{\partial y^2} = -\frac{d}{du}\left(\frac{df}{du}\right)\frac{\partial u}{\partial y} + \frac{d}{dv}\left(\frac{dg}{dv}\right)\frac{\partial v}{\partial y}$

$= \frac{d^2 f}{du^2} + \frac{d^2 g}{dv^2}$

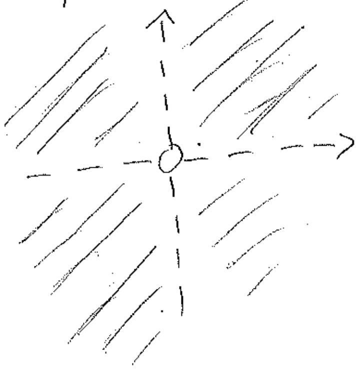
$\to \to$

portanto,

$$\frac{\partial^2 w}{\partial x^2} - 4\frac{\partial^2 w}{\partial y^2} = \cancel{4\frac{d^2 f}{du^2}} + \cancel{4\frac{d^2 g}{dv^2}} - 4\left(\cancel{\frac{d^2 f}{du^2}} + \cancel{\frac{d^2 g}{dv^2}}\right) = 0.$$

$$\{(x,y)\in\mathbb{R}^2 : xy\neq 0\} = \mathbb{R}^2 \setminus \{(x,y)\in\mathbb{R}^2 : x=0 \vee y=0\}$$

5a) $D_f = \{(x,y)\in\mathbb{R}^2 : xy\neq 0\}$



b) $$\frac{\partial f}{\partial x} = 1 + 2\left[(xy)^{-1}\right]'_x = 1 - 2(xy)^{-2}\cdot y$$

$$= 1 - \frac{2y}{x^2y^2} = 1 - \frac{2}{x^2y}$$

$$\frac{\partial f}{\partial y} = 4 + 2\left[(xy)^{-1}\right]'_y = 4 - 2(xy)^{-2}x$$

$$= 4 - \frac{2x}{x^2y^2} = 4 - \frac{2}{xy^2}$$

Pontos críticos:

$$\vec{\nabla} f = \vec{0} \Leftrightarrow \begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \\ ((x,y)\in D_f) \end{cases} \Leftrightarrow \begin{cases} 1 - \frac{2}{x^2y} = 0 \\ \cancel{4}^{2} - \frac{\cancel{2}^{1}}{xy^2} = 0 \\ (x\neq 0,\ y\neq 0) \end{cases} \Leftrightarrow \begin{cases} \frac{2}{x^2y} = 1 \\ \frac{1}{xy^2} = 2 \\ (\ldots) \end{cases}$$

$$\Leftrightarrow \begin{cases} y = \frac{2}{x^2} \\ 2xy^2 = 1 \\ (x\neq 0,\ y\neq 0) \end{cases} \Leftrightarrow \begin{cases} 2x\cdot\left(\frac{2}{x^2}\right)^2 = 1 \\ \text{———} \end{cases} \Leftrightarrow \begin{cases} 8/x^3 = 1 \\ \text{———} \end{cases}$$

$$\Leftrightarrow \begin{cases} y = \frac{1}{2} \\ x = 2 \end{cases}$$, único ponto crítico de f.

$$\frac{\partial^2 f}{\partial x^2} = \left(1 - \frac{2}{x^2y}\right)'_x = -2\left[(x^2y)^{-1}\right]'_x = 2\,(x^2y)^{-2}\cdot(x^2y)'_x$$

$$= 2\,\frac{2xy}{x^4y^2} = \frac{4}{x^3y} \Rightarrow \frac{\partial^2 f}{\partial x^2}\left(2,\tfrac{1}{2}\right) = \frac{4}{8\cdot\frac{1}{2}} = 1$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \left(1 - \frac{2}{x^2y}\right)'_y = -2\left[(x^2y)^{-1}\right]'_y = 2\,(x^2y)^{-2}\cdot(x^2y)'_y$$

$$= 2\,\frac{x^2}{x^4y^2} = \frac{2}{x^2y^2} \Rightarrow \frac{\partial^2 f}{\partial y\,\partial x}\left(2,\tfrac{1}{2}\right) = \frac{2}{4\cdot\frac{1}{4}} = 2$$

$$\frac{\partial^2 f}{\partial y^2} = \left(4 - \frac{2}{xy^2}\right)'_y = -2\left[(xy^2)^{-1}\right]'_y = 2\,(xy^2)^{-2}\cdot(xy^2)'_y$$

$$= 2\,\frac{2xy}{x^2y^4} = \frac{4}{xy^3} \Rightarrow \frac{\partial^2 f}{\partial y^2}\left(2,\frac{1}{2}\right) = \frac{4}{2\cdot\frac{1}{8}} = 16$$

Então

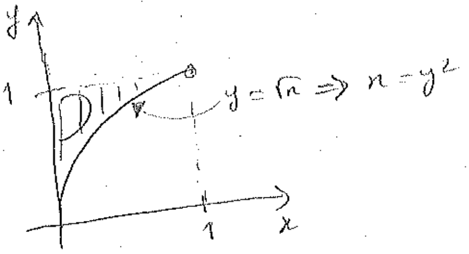
$$D^2 f\left(2,\frac{1}{2}\right) = \begin{bmatrix} 1 & 2 \\ 2 & 16 \end{bmatrix} \qquad a = 1 > 0\,,\ D = 16-4 = 12 > 0$$

logo $\left(2,\frac{1}{2}\right)$ é um mínimo.

<u>6</u> a)

D : $0 \le x \le 1$

$\sqrt{x} \le y \le 1$



b) D : $0 \le y \le 1$

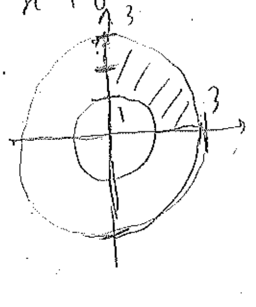
$0 \le x \le y^2$

$$\int_0^1\int_{\sqrt{x}}^1 \frac{1}{\sqrt{1+y^3}}\,dy\,dx = \int_0^1\int_0^{y^2}\frac{1}{\sqrt{1+y^3}}\,dx\,dy = \int_0^1 \underbrace{y^2}_{u'}\,(\underbrace{1+y^3}_{u})^{-1/2}\,dy$$

$$= \left[\frac{(1+y^3)^{-1/2+1}}{-1/2+1}\right]_0^1 = 2\left[2^{1/2} - 1^{1/2}\right] = 2\left(\sqrt{2}-1\right)$$

<u>7</u> $\frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$

a)

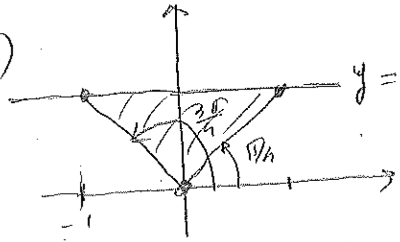


D : $1 \le r \le 3$

$0 \le \theta \le \pi/2$

$$\iint_D \frac{x}{x^2+y^2}\,dA = \int_0^{\pi/2}\int_1^3 \frac{\cos\theta}{\cancel{r}}\,\cancel{r}\,dr\,d\theta$$

→ →

b) 

$y = 1 \Leftrightarrow r\sin\theta = 1 \Leftrightarrow r = 1/\sin\theta$

$D: \ \pi/4 \le \theta \le 3\pi/4$

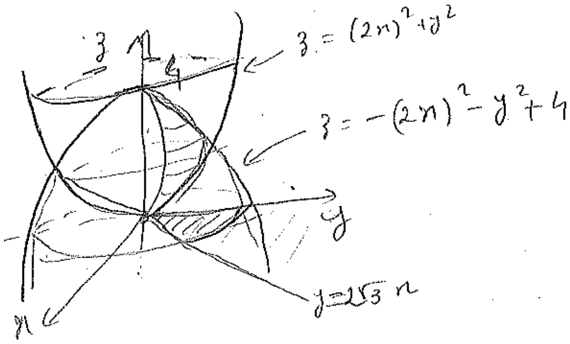
$\quad\ 0 \le r \le 1/\sin\theta$

$$\iint_D \frac{x}{x^2+y^2}\, dA = \int_{\pi/4}^{3\pi/4} \int_0^{\frac{1}{\sin\theta}} \frac{\cos\theta}{\cancel{r}}\, \cancel{r}\, dr\, d\theta .$$

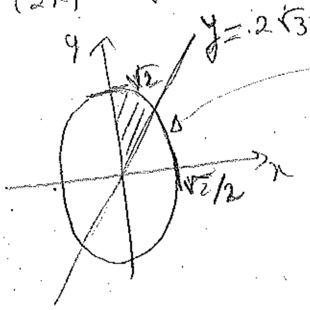
8

$$\text{Volume} = \iint_D -(2x)^2 - y^2 + 4 - \left((2x)^2 + y^2\right) dx\,dy$$

$$= \iint_D -2\left[(2x)^2 + y^2\right] + 4 \ dx\,dy$$



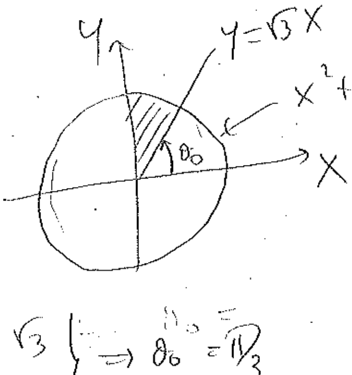
$D: \ (2x)^2 + y^2 = -(2x)^2 - y^2 + 4 \Leftrightarrow (2x)^2 + y^2 = 2$



Fazendo a mudança de coordenadas $\begin{cases} X = 2x \\ Y = y \end{cases}$, obtemos

$$\frac{\partial(x,y)}{\partial(X,Y)} = \begin{vmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{vmatrix} = \frac{1}{2}, \text{ donde } dx\,dy = \frac{1}{2}\, dX\,dY \text{ e}$$

$$\text{Volume} = \iint_{\tilde{D}} \left(-\overset{1}{\cancel{2}} \left[X^2 + Y^2\right] + \overset{2}{\cancel{4}} \right) \frac{1}{\cancel{2}}\, dX\,dY ,$$

$\tilde{D}$: 

$0 \le \theta \le \theta_0 = \pi/3$

$0 \le r \le \sqrt{2}$

$\left.\begin{array}{l} \operatorname{tg}\theta_0 = \sqrt{3} \\ , \theta_0 \in 1^{\circ}Q \end{array}\right\} \Rightarrow \theta_0 = \pi/3$

$\longrightarrow \longrightarrow$

$$\text{Volume} = \int_0^{\pi/3} \int_0^{\sqrt{2}} (-r^2+2)\, r \, dr\, d\theta$$

$$= \pi/3 \left[-\frac{r^4}{4} + r^2 \right]_0^{\sqrt{2}} = \pi/3 \left[-\frac{4}{4} + 2 \right] = \pi/3 .$$

9 Se (x^*, y^*) for um ponto crítico de f. Escrevendo

$$\nabla^2 f(x^*, y^*) = \begin{bmatrix} a & b \\ b & c \end{bmatrix},$$

Como, por hipótese, $\frac{\partial^2 f}{\partial x^2}(x^*, y^*) + \frac{\partial^2 f}{\partial y^2}(x^*, y^*) = 0 \Leftrightarrow a + c = 0 \Leftrightarrow c = -a$,

portanto

$$\det\left(\nabla^2 f(x^*, y^*)\right) = \begin{vmatrix} a & b \\ b & -a \end{vmatrix} = -a^2 - b^2 < 0,$$

onde não podemos obter igualdade porque, por hipótese, os pontos críticos são não degenerados.

Concluímos então que os pontos críticos, se existirem, têm de ser pontos de sela. Em particular, não há máximos nem mínimos.