EXAME 1ª ÉPOCA TMII 20-04-21

1

Se $(x,y) \neq (0,0)$

$$g_x = \left(\frac{x^2 \sin(4x) + y^2}{2x^2 + y^4} \right)'_x$$

$$= \frac{(2x \sin 4x + 4x^2 \cos 4x) \cdot (2x^2+y^4) - 4x(x^2 \sin 4x + y^2)}{(2x^2+y^4)^2}$$

$$g_x(0,0) = \lim_{h \to 0} \frac{g(h,0) - g(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left(\frac{h^2 \sin 4h + 0}{2h^2} \right)$$

$$= 2 \lim_{h \to 0} \frac{\sin 4h}{4h} = 2$$

$\underline{\underline{2}}$

a) $u = 2xy \quad , \quad v = \ln(xy)$

$$\frac{\partial w}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$

$$= \frac{\partial f}{\partial u} 2x + \frac{\partial f}{\partial v}\frac{x}{xy}$$

$$= 2x\frac{\partial f}{\partial u} + \frac{1}{y}\frac{\partial f}{\partial v}$$

b) $\dfrac{\partial^2 w}{\partial y \partial x} = \dfrac{\partial^2 w}{\partial x \partial y}$

$$= \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(2x\frac{\partial f}{\partial u} + \frac{1}{y}\frac{\partial f}{\partial v}\right)$$

$$= 2\frac{\partial f}{\partial u} + 2x\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\right)$$

$$+ \frac{1}{y}\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v}\right)$$

$$= 2\frac{\partial f}{\partial u} + 2x\left[\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u}\right)\frac{\partial v}{\partial x}\right]$$

$$+ \frac{1}{y}\left[\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= 2\frac{\partial f}{\partial u} + 2x\left[\frac{\partial^2 f}{\partial u^2}2y + \frac{\partial^2 f}{\partial v\partial u}\frac{y}{xy}\right]$$

$$+ \frac{1}{y}\left[\frac{\partial^2 f}{\partial u\partial v}2y + \frac{\partial^2 f}{\partial v^2}\frac{y}{xy}\right]$$

$$= 2\frac{\partial f}{\partial u} + 4xy\frac{\partial^2 f}{\partial u^2} + 4\frac{\partial^2 f}{\partial u \partial v}$$

$$+ \frac{1}{2y}\frac{\partial^2 f}{\partial v^2}$$

3

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Leftrightarrow \begin{cases} -2xe^{y} = 0 \\ e^{y}(y^2 - x^2) + e^{y} \cdot 2y = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \\ e^{y} y(y+2) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \\ y = 0 \vee y = -2 \end{cases}$$

Pontos críticos $= \{(0,0), (0,-2)\}$

$$\frac{\partial^2 f}{\partial x^2} = -2e^y$$

$$\frac{\partial^2 f}{\partial y \partial x} = -2x e^y$$

$$\frac{\partial^2 f}{\partial y^2} = \left(e^y \left(y^2 + 2y - x^2 \right) \right)'_y$$

$$= e^y \left(y^2 + 2y - x^2 + 2y + 2 \right)$$

$$= e^y \left(y^2 + 4y - x^2 + 2 \right)$$

$$H(0,0) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

$D = -4 < 0$

$\Longrightarrow (0,0)$ é ponto de sela

$$H(0,-2) = \begin{bmatrix} -2e^{-2} & 0 \\ 0 & -2e^{-2} \end{bmatrix}$$

$$D = 4e^{-4} > 0$$

$$\frac{\partial^2 f}{\partial x^2}(0,-2) = -2e^{-2} < 0$$

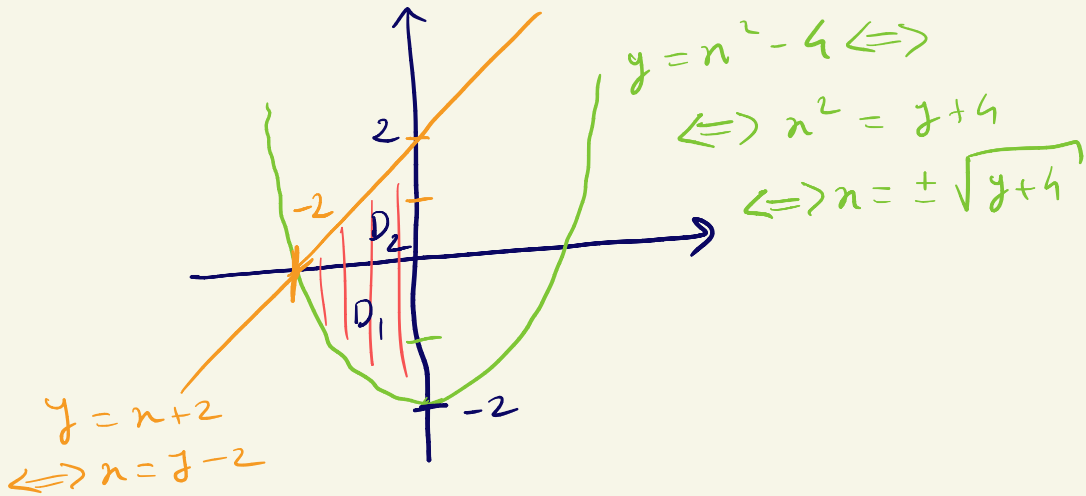
$\Rightarrow (0,-2)$ é um máximo.

4

a)

$D: -2 \leq x \leq 0$

$x^2 - 4 \leq y \leq x+2$



b) D_1 : $-2 \leq y \leq 0$

$$-\sqrt{y+4} \leq x \leq 0$$

D_2 : $0 \leq y \leq 2$

$$y-2 \leq x \leq 0$$

$$\iint_D 3x^2 y \, dy \, dx =$$

$$= \int_{-2}^{0} \int_{-\sqrt{y+4}}^{0} 3x^2 y \, dx \, dy$$

$$+ \int_{0}^{2} \int_{y-2}^{0} 3x^2 y \, dx \, dy$$

c)

$$\int_{-2}^{0} \int_{x^2-4}^{x+2} 3x^2 y \, dy \, dx =$$

$$= 3 \int_{-2}^{0} x^2 \left[\frac{y^2}{2} \right]_{x^2-4}^{x+2} dx$$

$$= \frac{3}{2} \int_{-2}^{0} x^2 \left((x+2)^2 - (x^2-4)^2 \right) dx$$

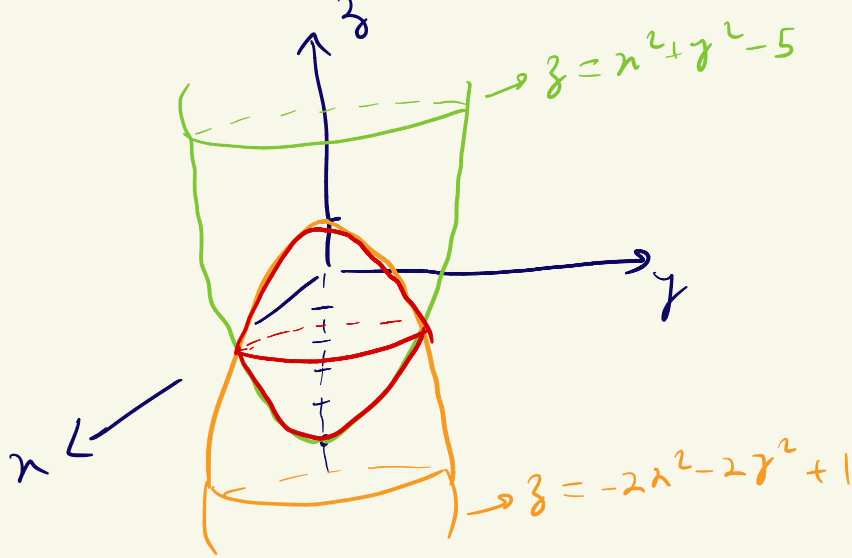
$$= \frac{3}{2}\int_{-2}^{0} x^2\Big(\underbrace{x^2+4x+4-x^4+8x^2-16}_{-x^4+9x^2+4x-12}\Big)\,dx$$

$$= \frac{3}{2}\int_{-2}^{0} -x^6+9x^4+4x^3-12x^2 \;dx$$

$$= \frac{3}{2}\left[-\frac{x^7}{7}+9\frac{x^5}{5}+x^4-12\frac{x^3}{3}\right]_{-2}^{0}$$

$$= -\frac{3}{2}\left[-\frac{(-2)^7}{7}+9\frac{(-2)^5}{5}+(-2)^4-4(-2)^3\right]$$

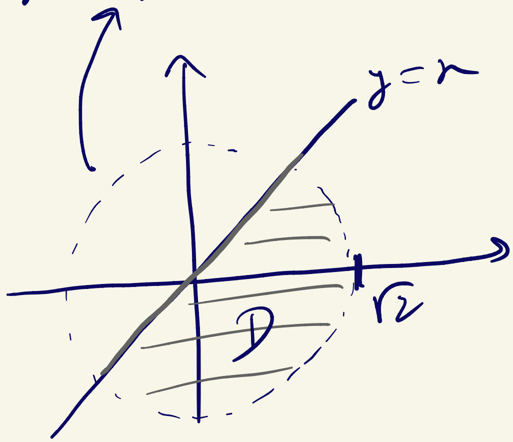
5
=



$D: x^2+y^2-5 = -2x^2-2y^2+1 \quad , \quad x \geq y$

$\iff 3x^2+3y^2 = 6 \quad , \quad x \geq y$

$\iff x^2+y^2 = 2 \quad , \quad x \geq y$



$D: \quad 0 \leq r \leq \sqrt{2}$

$\underbrace{-\pi/2 - \pi/4}_{-3\pi/4} \leq \theta \leq \pi/4$

$$\text{volume} = \iint_D -2x^2-2y^2+1-(x^2+y^2-5)\, dA$$

$$= \iint_D -3(x^2+y^2)+6 \; dA$$

$$= \int_0^{\sqrt{2}} \int_{-3\pi/4}^{\pi/4} \left(-3r^2+6\right) r \, d\theta \, dr$$

$$= \pi \int_0^{\sqrt{2}} -3r^3+6r \; dr$$

$$= \pi \left[-3\frac{r^4}{4} + 6\frac{r^2}{2} \right]_0^{\sqrt{2}}$$

$$= \pi \left[-\frac{3}{4} 4 + 3 \times 2 \right]$$

$$= 3\pi$$

6

diagonal $= d = 1 \Longrightarrow$

$\Longrightarrow 1 = \sqrt{x^2+y^2+z^2}$

$\Longrightarrow 1 = x^2+y^2+z^2$

$\Longrightarrow z = \sqrt{1-x^2-y^2}$

$$\begin{cases} \text{Volume} = V = xyz \\ d = 1 \end{cases} \Longrightarrow$$

$\Longrightarrow V = xy\sqrt{1-x^2-y^2}$

$$\begin{cases} V_x = 0 \\ V_y = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y\sqrt{1-x^2-y^2} + \frac{1}{2}xy\left(1-x^2-y^2\right)^{-1/2} \cdot (-2x) = 0 \\ x\sqrt{1-x^2-y^2} + \frac{1}{2}xy\left(1-x^2-y^2\right)^{-1/2}(-2y) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \dfrac{y}{\sqrt{1-x^2-y^2}}\left(1-x^2-y^2-x^2\right) = 0 \\ \dfrac{x}{\sqrt{1-x^2-y^2}}\left(1-x^2-y^2-y^2\right) = 0 \end{cases}$$

Como $x=0 \vee y=0 \Rightarrow V=0$,

i.e., um volume mínimo,

consideramos $x \neq 0$ e $y \neq 0$,

donde

$$\Rightarrow \begin{cases} 1 - 2x^2 - y^2 = 0 \\ 1 - x^2 - 2y^2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} -1 + 0 + 3y^2 = 0 \\ \text{———} \end{cases}$$

$$\Leftrightarrow \begin{cases} y = \pm\sqrt{1/3} \\ 1 - x^2 - 2\,\frac{1}{3} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = \pm\sqrt{1/3} \\ x = \pm\sqrt{\frac{1}{3}} \end{cases}$$

$\therefore$ O volume máximo desejado é atingido quando $x = y = \sqrt{1/3}$

$$\Rightarrow z = \sqrt{1 - 1/3 - 1/3} = \sqrt{1/3}\,.$$